

A new error bound for linear complementarity problems for B -matrices

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Abstract

A new error bound for the linear complementarity problem is given when the involved matrix is a B -matrix. It is shown that this bound is sharper than some previous bounds [C.Q. Li, Y.T. Li. Note on error bounds for linear complementarity problems for B -matrices, *Applied Mathematics Letters*, 57:108-113, 2016] and [C.Q. Li, Y.T. Li. Weakly chained diagonally dominant B -matrices and error bounds for linear complementarity problems, to appear in Numer. Algor.].

Keywords: Error bound, Linear complementarity problem, B -matrix

1. Introduction

Linear complementarity problem $LCP(M, q)$ is to find a vector $x \in R^n$ such that

$$x \geq 0, Mx + q \geq 0, (Mx + q)^T x = 0, \quad (1)$$

where $M = [m_{ij}] \in R^{n \times n}$ and $q \in R^n$. The $LCP(M, q)$ has various applications in the Nash equilibrium point of a bimatrix game, the contact problem and the free boundary problem for journal bearing, for details, see [4, 5, 17].

It is well-known that the $LCP(M, q)$ has a unique solution for any $q \in R^n$ if and only if M is a P -matrix [5]. Here, a matrix $M \in R^{n \times n}$ is called a P -matrix if all its principal minors are positive [6]. In [3], Chen and Xiang gave the following error bound of the $LCP(M, q)$ when M is a P -matrix:

$$\|x - x^*\|_\infty \leq \max_{d \in [0, 1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty,$$

where x^* is the solution of the $LCP(M, q)$, $r(x) = \min\{x, Mx + q\}$, $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. If M satisfies certain structure, then some bounds of

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$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ can be derived; for details, see [2, 7, 8, 10, 14] and references therein.

When M is a B -matrix introduced by Peña in [6] as a subclass of P -matrices, García-Esnaola and Peña in [10] presented the following upper bound which is only related with the entries of M . Here a real matrix $M = [m_{ij}] \in R^{n \times n}$ is called a B -matrix [6] if for each $i \in N = \{1, 2, \dots, n\}$,

$$\sum_{k \in N} m_{ik} > 0, \text{ and } \frac{1}{n} \left(\sum_{k \in N} m_{ik} \right) > m_{ij}, \text{ for any } j \in N \text{ and } j \neq i. \quad (2)$$

Theorem 1. [10, Theorem 2.2] *Let $M = [m_{ij}] \in R^{n \times n}$ be a B -matrix with the form*

$$M = B^+ + C, \quad (3)$$

where

$$B^+ = [b_{ij}] = \begin{bmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{bmatrix}, \quad (4)$$

and $r_i^+ = \max\{0, m_{ij} | j \neq i\}$. Then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{n-1}{\min\{\beta, 1\}}, \quad (5)$$

where $\beta = \min_{i \in N} \{\beta_i\}$ and $\beta_i = b_{ii} - \sum_{j \neq i} |b_{ij}|$.

As shown in [15], if the diagonal dominance of B^+ is weak, i.e.,

$$\beta = \min_{i \in N} \{\beta_i\} = \min_{i \in N} \left\{ b_{ii} - \sum_{j \neq i} |b_{ij}| \right\}$$

is small, then the bound (5) may be very large when M is a B -matrix, which leads to that the estimate in (5) is always inaccurate, for details, see [15, 16]. To improve the bound (5), Li and Li [15] gave the following bound for $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a B -matrix.

Theorem 2. [15, Theorem 4] *Let $M = [m_{ij}] \in R^{n \times n}$ be a B -matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of (4). Then*

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \sum_{i=1}^n \frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right), \quad (6)$$

where $\bar{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}| l_i(B^+)$, $l_k(B^+) = \max_{k \leq i \leq n} \left\{ \frac{1}{|b_{ii}|} \sum_{j=k, j \neq i}^n |b_{ij}| \right\}$ and

$$\prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right) = 1 \text{ if } i = 1.$$

Very recently, when M is a weakly chained diagonally dominant B -matrix, Li and Li [16] gave a bound for $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$. This bound holds true for the case that M is a B -matrix because a B -matrix is a weakly chained diagonally dominant B -matrix [16].

Theorem 3. [16, Corollary 1] *Let $M = [m_{ij}] \in R^{n \times n}$ be a B -matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of (4). Then*

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \sum_{i=1}^n \left(\frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} \right),$$

where $\tilde{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}| > 0$ and $\prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} = 1$ if $i = 1$.

In this paper, we also focus on the error bound for the LCP(M, q), and gave a new bound for $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a B -matrix. It is shown that this bound is more effective to estimate $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ than that in Theorem 1, and sharper than those in Theorems 2 and 3.

2. Main results

We first recall some definitions. A matrix $A = [a_{ij}] \in C^{n \times n}$ is called a strictly diagonally dominant (SDD) matrix if for each $i \in N$, $|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^n |a_{ij}|$.

It is well-known that an SDD matrix is nonsingular [1]. A matrix $A = [a_{ij}]$ is called a Z -matrix if $a_{ij} \leq 0$ for any $i \neq j$, and a nonsingular M -matrix if A is a Z -matrix with A^{-1} being nonnegative [1]. Next, several lemmas which will be used later are given.

Lemma 4. [18, Theorem 3.2] *Let $A = [a_{ij}] \in R^{n \times n}$ be an SDD M -matrix. Then*

$$\|A^{-1}\|_\infty \leq \sum_{i=1}^n \left(\frac{1}{a_{ii}(1 - u_i(A)l_i(A))} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(A)l_j(A)} \right),$$

where $u_i(A) = \frac{1}{|a_{ii}|} \sum_{j=i+1}^n |a_{ij}|$, $l_k(A) = \max_{k \leq i \leq n} \left\{ \frac{1}{|a_{ii}|} \sum_{\substack{j=k, \\ j \neq i}}^n |a_{ij}| \right\}$ and

$$\prod_{j=1}^{i-1} \frac{1}{1 - u_j(A)l_j(A)} = 1 \text{ if } i = 1.$$

Lemma 5. [15, Lemma 3] *Let $\gamma > 0$ and $\eta \geq 0$. Then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}} \quad (7)$$

and

$$\frac{\eta x}{1-x+\gamma x} \leq \frac{\eta}{\gamma}. \quad (8)$$

Lemma 6. [16, Lemma 5] Let $A = [a_{ij}] \in R^{n \times n}$ with

$$a_{ii} > \sum_{j=i+1}^n |a_{ij}| \text{ for each } i \in N.$$

Then for any $x_i \in [0, 1]$, $i \in N$,

$$\frac{1-x_i+a_{ii}x_i}{1-x_i+a_{ii}x_i-\sum_{j=i+1}^n |a_{ij}|x_i} \leq \frac{a_{ii}}{a_{ii}-\sum_{j=i+1}^n |a_{ij}|}.$$

The main result is given as follows.

Theorem 7. Let $M = [m_{ij}] \in R^{n \times n}$ be a B -matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of (4). Then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \sum_{i=1}^n \frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}, \quad (9)$$

where $\bar{\beta}_i$ is defined in Theorem 2 and $\prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = 1$ if $i = 1$.

Proof. Let $M_D = I - D + DM$. Then

$$M_D = I - D + DM = I - D + D(B^+ + C) = B_D^+ + C_D,$$

where $B_D^+ = I - D + DB^+ = [b_{ij}]$. Similar to the proof of Theorem 2.2 in [10], we can obtain that B_D^+ is an SDD M -matrix with positive diagonal elements and $C_D = DC$, and that

$$\|M_D^{-1}\|_{\infty} \leq \|(I + (B_D^+)^{-1}C_D)^{-1}\|_{\infty} \|(B_D^+)^{-1}\|_{\infty} \leq (n-1)\|(B_D^+)^{-1}\|_{\infty}. \quad (10)$$

By Lemma 4,

$$\|(B_D^+)^{-1}\|_{\infty} \leq \sum_{i=1}^n \left(\frac{1}{(1-d_i+b_{ii}d_i)(1-u_i(B_D^+)l_i(B_D^+))} \prod_{j=1}^{i-1} \frac{1}{1-u_j(B_D^+)l_j(B_D^+)} \right),$$

where

$$u_i(B_D^+) = \frac{\sum_{j=i+1}^n |b_{ij}|d_i}{1-d_i+b_{ii}d_i}, \text{ and } l_k(B_D^+) = \max_{k \leq i \leq n} \left\{ \frac{\sum_{\substack{j=k, \\ j \neq i}}^n |b_{ij}|d_i}{1-d_i+b_{ii}d_i} \right\}.$$

By Lemma 5, we can easily get that for each $k \in N$,

$$l_k(B_D^+) \leq \max_{k \leq i \leq n} \left\{ \frac{1}{b_{ii}} \sum_{\substack{j=k, \\ j \neq i}}^n |b_{ij}| \right\} = l_k(B^+) < 1, \quad (11)$$

and that for each $i \in N$,

$$\begin{aligned} \frac{1}{(1 - d_i + b_{ii}d_i)(1 - u_i(B_D^+)l_i(B_D^+))} &= \frac{1}{1 - d_i + b_{ii}d_i - \sum_{j=i+1}^n |b_{ij}|d_i l_i(B_D^+)} \\ &\leq \frac{1}{\min \left\{ b_{ii} - \sum_{j=i+1}^n |b_{ij}|l_i(B^+), 1 \right\}} \\ &= \frac{1}{\min \{\bar{\beta}_i, 1\}}. \end{aligned} \quad (12)$$

Furthermore, by Lemma 6,

$$\frac{1}{1 - u_i(B_D^+)l_i(B_D^+)} = \frac{1 - d_i + b_{ii}d_i}{1 - d_i + b_{ii}d_i - \sum_{j=i+1}^n |b_{ij}|d_i l_i(B_D^+)} \leq \frac{b_{ii}}{\bar{\beta}_i}. \quad (13)$$

By (11), (11), (12) and (13), we have

$$\|(B_D^+)^{-1}\|_\infty \leq \frac{1}{\min \{\bar{\beta}_1, 1\}} + \sum_{i=2}^n \frac{1}{\min \{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}. \quad (14)$$

The conclusion follows from (10) and (14). \square

The comparisons of the bounds in Theorems 2, 3 and 7 are established as follows.

Theorem 8. Let $M = [m_{ij}] \in R^{n \times n}$ be a B -matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of (4). Let $\bar{\beta}_i$ and $\tilde{\beta}_i$ be defined in Theorems 2 and 3 respectively. Then

$$\begin{aligned} \sum_{i=1}^n \frac{n-1}{\min \{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} &\leq \sum_{i=1}^n \frac{n-1}{\min \{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right) \\ &\leq \sum_{i=1}^n \left(\frac{n-1}{\min \{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} \right). \end{aligned}$$

Proof. Note that

$$\tilde{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}|, \quad \bar{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}|l_i(B^+)$$

and $l_k(B^+) = \max_{k \leq i \leq n} \left\{ \frac{1}{|b_{ii}|} \sum_{\substack{j=k, \\ j \neq i}}^n |b_{ij}| \right\} < 1$. Hence, for each $i \in N$, $\tilde{\beta}_i \leq \bar{\beta}_i$ and

$$\frac{1}{\min\{\tilde{\beta}_i, 1\}} \geq \frac{1}{\min\{\bar{\beta}_i, 1\}}. \quad (15)$$

Meantime, for $j = 1, 2, \dots, n-1$,

$$1 + \frac{1}{\beta_j} \sum_{k=j+1}^n |b_{jk}| \leq 1 + \frac{1}{\tilde{\beta}_j} \sum_{k=j+1}^n |b_{jk}| = \frac{1}{\tilde{\beta}_j} \left(\tilde{\beta}_j + \sum_{k=j+1}^n |b_{jk}| \right) = \frac{b_{jj}}{\tilde{\beta}_j}. \quad (16)$$

By (15) and (16), we have

$$\sum_{i=1}^n \frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\tilde{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right) \leq \sum_{i=1}^n \left(\frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} \right). \quad (17)$$

Moreover, for $j = 1, 2, \dots, n-1$,

$$\begin{aligned} \frac{b_{jj}}{\tilde{\beta}_j} &= \prod_{j=1}^{i-1} \frac{b_{jj} - \sum_{k=j+1}^n |b_{jk}| l_j(B^+) + \sum_{k=j+1}^n |b_{jk}| l_j(B^+)}{\tilde{\beta}_j} \\ &= \frac{\tilde{\beta}_j + \sum_{k=j+1}^n |b_{jk}| l_j(B^+)}{\tilde{\beta}_j} \\ &= \left(1 + \frac{\sum_{k=j+1}^n |b_{jk}| l_j(B^+)}{\tilde{\beta}_j} \right) \\ &\leq \left(1 + \frac{\sum_{k=j+1}^n |b_{jk}|}{\tilde{\beta}_j} \right), \end{aligned}$$

this implies

$$\sum_{i=1}^n \frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} \leq \sum_{i=1}^n \frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\tilde{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right). \quad (18)$$

The conclusion follows from (17) and (18). \square

Example 1. Consider the family of B -matrices in [15]:

$$M_k = \begin{bmatrix} 1.5 & 0.5 & 0.4 & 0.5 \\ -0.1 & 1.7 & 0.7 & 0.6 \\ 0.8 & -0.1 \frac{k}{k+1} & 1.8 & 0.7 \\ 0 & 0.7 & 0.8 & 1.8 \end{bmatrix},$$

where $k \geq 1$. Then $M_k = B_k^+ + C_k$, where

$$B_k^+ = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1\frac{k}{k+1} - 0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{bmatrix}.$$

By Theorem 1 (Theorem 2.2 in [10]), we have

$$\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_\infty \leq \frac{4-1}{\min\{\beta, 1\}} = 30(k+1).$$

It is obvious that

$$30(k+1) \rightarrow +\infty, \text{ when } k \rightarrow +\infty.$$

By Theorem 3, we have

$$\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_\infty \leq \sum_{i=1}^4 \left(\frac{3}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} \right) \approx 15.2675.$$

By Theorem 2, we have

$$\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty \leq \frac{2.97(90k+91)(190k+192) + 6.24(100k+101)^2}{0.99(90k+91)^2},$$

and

$$\frac{2.97(90k+91)(190k+192) + 6.24(100k+101)^2}{0.99(90k+91)^2} < 15.2675, \text{ for any } k \geq 1.$$

By Theorem 7, we have

$$\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty \leq \frac{2.97(90k+91)(190k+191) + 5.97(100k+100)^2}{0.99(90k+91)^2},$$

and

$$\begin{aligned} & \frac{2.97(90k+91)(190k+191) + 5.97(100k+100)^2}{0.99(90k+91)^2} \\ & < \frac{2.97(90k+91)(190k+192) + 6.24(100k+101)^2}{0.99(90k+91)^2}. \end{aligned}$$

In particular, when $k = 1$,

$$\frac{2.97(90k+91)(190k+191) + 5.97(100k+100)^2}{0.99(90k+91)^2} \approx 13.6777,$$

and

$$\frac{2.97(90k+91)(190k+192) + 6.24(100k+101)^2}{0.99(90k+91)^2} \approx 14.1044.$$

When $k = 2$,

$$\frac{2.97(90k + 91)(190k + 191) + 5.97(100k + 100)^2}{0.99(90k + 91)^2} \approx 13.7110,$$

and

$$\frac{2.97(90k + 91)(190k + 192) + 6.24(100k + 101)^2}{0.99(90k + 91)^2} \approx 14.1079.$$

In these two cases, the bounds in (5) are equal to 60 ($k = 1$) and 90 ($k = 2$) respectively. This example shows that the bound in Theorem 7 is sharper than those in Theorems 1, 2 and 3.

3. Conclusions

In this paper, we give a new bound for $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a B -matrix, which improves the bounds obtained in [10, 15, 16].

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